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LETTER TO THE EDITOR

Ground state of the quantum symmetric finite-size XXZ spin chain with anisotropy parameter $\Delta = \frac{1}{2}$*

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Abstract. We find an analytic solution of the Bethe Ansatz equations for the special case of a finite XXZ spin chain with free boundary conditions and with a complex surface field which provides for $U_q(sl(2))$ symmetry of the Hamiltonian. More precisely, we find one nontrivial solution, corresponding to the ground state of the system with anisotropy parameter $\Delta = \frac{1}{2}$ corresponding to $q^3 = -1$.

It is widely accepted that the Bethe Ansatz equations (BAE) for an integrable quantum spin chain can be solved analytically only in the thermodynamic limit or for a small number of spin waves or short chains. In this letter, however, we have managed to find a special solution of the BAE for a spin chain of arbitrary length N with $\frac{N}{2}$ spin waves.

It is well known (see, for example [1] and references therein) that there is a correspondence between the Q-state Potts models and the ice-type models with anisotropy parameter $\Delta = \frac{\sqrt{Q}}{2}$. The coincidence in the spectrum of an N-site self-dual Q-state quantum Potts chain with free ends with a part of the spectrum of the $U_q(sl(2))$ symmetrical $2N$-site XXZ Hamiltonian (1) is to some extent a manifestation of this correspondence:

$$H_{xxz} = \sum_{n=1}^{N-1} \left\{ \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{q + q^{-1}}{4} \sigma_n^z \sigma_{n+1}^z + \frac{q - q^{-1}}{4} (\sigma_n^z - \sigma_{n+1}^z) \right\} \tag{1}$$

where $\Delta = (q + q^{-1})/2$. This Hamiltonian was considered by Alcaraz *et al* [1] and its $U_q(sl(2))$ symmetry was described by Pasqier and Saleur [2]. The family of commuting transfer matrices that commute with H_{xxz} was constructed by Sklyanin [3] incorporating a method of Cherednik [4].

Baxter's T–Q equation for the case under consideration can be written as [5]

$$t(u)Q(u) = \phi\left(u + \frac{\eta}{2}\right) Q(u - \eta) + \phi\left(u - \frac{\eta}{2}\right) Q(u + \eta) \tag{2}$$

where $q = \exp i\eta$, $\phi(u) = \sin 2u \sin^{2N} u$ and $t(u) = \sin 2u T(u)$. The $Q(u)$ are eigenvalues of Baxter's auxilary matrix $\hat{Q}(u)$, where $\hat{Q}(u)$ commutes with the transfer matrix $\hat{T}(u)$. The

* Dedicated to Rodney Baxter on the occasion of his 60th birhday.

eigenvalue $Q(u)$ corresponding to an eigenvector with $M = \frac{N}{2} - S_z$ reversed spins has the form

$$Q(u) = \prod_{m=1}^{M} \sin(u - u_m)\sin(u + u_m).$$

Equation (2) is equivalent to the BAE [6]

$$\left[\frac{\sin(u_k + \eta/2)}{\sin(u_k - \eta/2)}\right]^{2N} = \prod_{m\neq k}^{M} \frac{\sin(u_k - u_m + \eta)\sin(u_k + u_m + \eta)}{\sin(u_k - u_m - \eta)\sin(u_k + u_m - \eta)}. \qquad (3)$$

In a recent article [7] Belavin and Stroganov argued that the criteria for the above-mentioned correspondence is the existence of a second trigonometric solution for Baxter's *T–Q* equation and it was shown that in the case $\eta = \frac{\pi}{4}$ the spectrum of H_{xxz} contains the spectrum of the Ising model. In this letter we limit ourselves to the case $\eta = \frac{\pi}{3}$. This case is in some sense trivial since for this value of η, H_{xxz} corresponds to the one-state Potts model. We find only one eigenvalue $T_0(u)$ of the transfer matrices $\hat{T}(u)$ when Baxter's equation (2) has two independent trigonometric solutions. Solving for $T(u) = T_0(u)$ analytically we find a trigonometric polynomial $Q_0(u)$, the zeros of which satisfy the BAE (3). The number of spin waves is equal to $M = \frac{N}{2}$. The corresponding eigenstate is the ground state of H_{xxz} with eigenvalue $E_0 = \frac{3}{2}(1 - N)$, as numerically discovered by Alcaraz *et al* [1].

When does a second independent periodic solution exist? This question was considered in [7]. Here we use a variation more convenient for our goal (see also [8]).

Let us consider the *T–Q* equation (2) for $\eta = \frac{\pi}{L}$, where $L \geqslant 3$ is an integer. Let us fix a sequence of spectral parameter values $v_k = v_0 + \eta k$, where k are integers and write $\phi_k = \phi(v_k - \eta/2)$, $Q_k = Q(v_k)$ and $t_k = t(v_k)$. The functions $\phi(u)$, $Q(u)$ and $t(u)$ are periodic with period π. Consequently, the sequences we have introduced are also periodic with period L, i.e., $\phi_{k+L} = \phi_k$, etc.

Setting $u = v_k$ in (2) gives the linear system

$$t_k Q_k = \phi_{k+1} Q_{k-1} + \phi_k Q_{k+1}. \qquad (4)$$

The matrix of coefficients for this system has a tridiagonal form. Taking $v_0 \neq \frac{\pi m}{2}$, where m is an integer, we have $\phi_k \neq 0$ for all k.

It is straightforward to calculate the determinant of the $L - 2 \times L - 2$ minor obtained by deleting the two left-most columns and two lower-most rows. It is equal to the product $-\phi_1^2\phi_2\phi_3 \ldots \phi_{L-1}$, which is nonzero, hence the rank of M cannot be less than $L - 2$. Here we are interested in the case when the rank of M is precisely $L - 2$ and we have two linearly independent solutions for equation (4). Let us consider the three simplest cases $L = 3$, 4 and 5. The parameter η is equal to $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$ respectively.

For $L = 3$ the rank of M is unity and we immediately get $t_0 = -\phi_2$, $t_1 = -\phi_0$ and $t_2 = -\phi_1$. Returning to the functional form, we can write

$$T_0(u) = \frac{t_0(u)}{\sin 2u} = \frac{-\phi(u + \frac{\pi}{2})}{\sin 2u} = \cos^{2N} u. \qquad (5)$$

This is the unique eigenvalue of the transfer matrix for which the *T–Q* equation has two independent periodic solutions. It is well known (see, for example, [6]) that the eigenvalues of H_{xxz} are related to the eigenvalues $t(u)$ by

$$E = -\cos\eta(N + 2 - \tan^2\eta) + \sin\eta\frac{t'(\frac{\eta}{2})}{t(\frac{\eta}{2})}.$$

For the eigenstate corresponding to eigenvalue (5) we obtain $E_0 = \frac{3}{2}(1 - N)$. This is the ground state energy which was discovered by Alcaraz *et al* [1] numerically.

Below we find all solutions of Baxter's *T*–*Q* equation corresponding to $T(u) = T_0(u)$. Zeros of these solutions satisfy the BAE (3). In particular, we find $Q_0(x)$ corresponding to physical Bethe state.

For $L = 4$, deleting the second row and the forth column of M we obtain a minor with determinant $-\phi_0\phi_3(t_0 + t_2)$. It is zero when $t_2 = -t_0$, i.e., $t(u + \frac{\pi}{2}) = -t(u)$. Considering the other minors we obtain the functional equation

$$t\left(u + \frac{\pi}{8}\right) t\left(u - \frac{\pi}{8}\right) = \phi\left(u + \frac{\pi}{4}\right)\phi\left(u - \frac{\pi}{4}\right) - \phi(u)\phi\left(u + \frac{\pi}{2}\right).$$

This functional equation was used in [7] to find $t(u)$ and show that this part of the spectrum of H_{xxz} coincides with the Ising model. It would be interesting to find a corresponding $Q(u)$.

Lastly, for $L = 5$, minor M_{35} (the third row and the fifth column are deleted) has determinant $\phi_0\phi_4(t_0t_1 + \phi_1t_3 - \phi_0\phi_2)$. Setting this to zero we have

$$t(u)t\left(u + \frac{\pi}{5}\right) + \phi\left(u + \frac{\pi}{10}\right) t\left(u + \frac{3\pi}{5}\right) - \phi\left(u - \frac{\pi}{10}\right)\phi\left(u + \frac{3\pi}{10}\right) = 0. \tag{6}$$

It is not difficult to check that in this case all 4×4 minors have zero determinant and that the rank of M is 3. Thus we have two independent periodic solutions of Baxter's *T*–*Q* equation.

Note that this functional relation coincides with the Baxter–Pearce relation for the hard hexagon model [9]. The connection between (6) and a special value of the rank of the matrix of coefficients for system (4) was remarked upon in [10] by Andrews *et al* (see also [8]).

For general L we obtain the same truncated functional relations that have been obtained in [7] with the same assumptions. Note that for the ABF models [10], which are a generalization of the hard hexagon model, the truncated functional relations have been proved by Behrend *et al* [11].

We now consider the solution of Baxter's equation for $\eta = \frac{\pi}{3}$ and $T = T_0$. For $\eta = \frac{\pi}{3}$ and transfer-matrix eigenvalue $T_0(u) = \cos^{2N} u$, the *T*–*Q* equation (2) reduces to

$$\phi\left(u + \frac{3\eta}{2}\right) Q(u) + \phi\left(u - \frac{\eta}{2}\right) Q(u + \eta) + \phi\left(u + \frac{\eta}{2}\right) Q(u - \eta) = 0.$$

This equation can be rewritten as

$$f(v) + f\left(v + \frac{2\pi}{3}\right) + f\left(v + \frac{4\pi}{3}\right) = 0 \tag{7}$$

where $f(v) = \sin v \cos^{2N}(v/2)Q(v/2)$ has period 2π. The trigonometric polynomial $f(v)$ is an odd function, so it can be written

$$f(v) = \sum_{k=1}^{K} c_k \sin kv \tag{8}$$

where K is the degree of $f(v)$. Then equation (7) is equivalent to $c_{3m} = 0$, $m \in Z$.

The condition that $f(v)$ be divisible by $\sin v \cos^{2N}(v/2)$ is equivalent to

$$\left(\frac{\mathrm{d}}{\mathrm{d}v}\right)^i f(v)|_{v=\pi} = 0 \qquad i = 0, 1, \ldots, 2N.$$

For even i this condition is immediate, whereas for $i = 2j - 1$ we use (8) to obtain

$$\sum_{k=1, k\neq 3m}^{K} (-1)^k c_k k^{2j-1} = 0 \qquad j = 1, 2, \ldots, N. \tag{9}$$

Our problem is thus to find $\{c_k\}$ satisfying the last equation. This problem is a special case of a more general problem which can be formulated as follows. Given a set of different complex numbers $X = \{x_1, x_2, \ldots, x_I\}$ we seek another complex set $B = \{\beta_1, \beta_2, \ldots, \beta_I\}$ where $\beta_i \neq 0$ for some i, so that

$$\sum_{i=1}^{I} \beta_i P(x_i) = 0 \tag{10}$$

for any polynomial $P(x)$ of degree not more than $N - 1$. It is clear that for $I \leqslant N$ the system B does not exist. If $\beta_1 \neq 0$, for example, the product $(x - x_2)(x - x_3) \ldots (x - x_I)$ provides a counter-example.

Let $I = N + 1$. We try the polynomials

$$P_r = \prod_{i=1, i\neq r,}^{N} (x - x_i) \qquad r = 1, 2, \ldots, N. \tag{11}$$

Condition (10) gives $\beta_r P_r(x_r) + \beta_I P_r(x_I) = 0$ and we immediately obtain

$$\beta_r = \text{const} \prod_{i=1, i\neq r}^{N+1} (x_r - x_i)^{-1} \tag{12}$$

which is a solution because the system (11) forms a basis of the linear space of $N - 1$ degree polynomials. So for $I = N+1$ we have a unique solution (up to an arbitrary nonzero constant) given by (12). It is easy to show that for $I = N + \nu$ we obtain a ν-dimensional linear space of solutions.

Returning to (9) we consider $N = 2n$, n a positive integer. Fix $I = N + 1 = 2n + 1$. The degree K becomes $3n + 1$. It is convenient to use a new index κ, where $|\kappa| \leqslant n$ and $k = |3\kappa + 1|$. Equation (9) can be rewritten as

$$\sum_{\kappa=-n}^{n} \beta_\kappa (3\kappa + 1)^{2(j-1)} = 0 \qquad j = 1, 2, \ldots, N$$

where we use new unknowns $\beta_\kappa = (-1)^\kappa c_{|3\kappa+1|} |3\kappa + 1|$ instead of c_k. Using (12) and (8) we obtain the function $f(v)$

$$f(v) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n + \frac{2}{3}}{n - \kappa} \binom{2n - \frac{2}{3}}{n + \kappa} \sin(3\kappa + 1)v. \tag{13}$$

We recall that the solution of Baxter's *T*–*Q* equation for $T(u) = T_0(u)$ is given by

$$Q_0(u) = \frac{f(2u)}{(\sin 2u \cos^{2N} u)} \tag{14}$$

and its zeros $\{u_k\}$ satisfy the BAE (3). In a similar manner we have obtained the second independent solution which we have used to find the first η-derivative of the transfer-matrix eigenvalue [12].

Another way to derive the above solution is to observe that the function $f(v)$ satisfies a simple second-order linear differential equation. Indeed, it is easily seen that the functions $F^+(x)$ and $F^-(x)$, where

$$F^+(x) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n + \frac{2}{3}}{n - \kappa} \binom{2n - \frac{2}{3}}{n + \kappa} x^{\kappa + \frac{1}{3}} \qquad \text{and} \qquad F^-(x) = F^+\left(\frac{1}{x}\right)$$

are the two linearly independent solutions of the differential equation

$$\{((\theta + n)^2 - \tfrac{1}{9})x^{-1} + (\theta - n)^2 - \tfrac{1}{9}\}F^+ = 0 \tag{15}$$

where $\theta = x\frac{\mathrm{d}}{\mathrm{d}x}$ (up to a change of variables this is just the standard hypergeometric differential equation, and in fact $F^+(x) = \text{const}F(-2n, \frac{2}{3} - 2n, \frac{5}{3}, -x)x^{1/3-n}$). Now the fact that there is a combination $f(v)$ of $F^+(e^{3iv})$ and $F^-(e^{3iv})$ which vanishes to order $2N+1$ at $v = \pi$ follows immediately from the fact that $x = -1$ is a singular point of the differential equation (15) and that the indicial equation at this point has roots 0 and $2n + 1$. In terms of the variable v, equation (15) becomes

$$\frac{d^2 f}{dv^2} + 6n \tan\left(\frac{3v}{2}\right)\frac{df}{dv} + (1 - 9n^2)f = 0.$$

The zeros of $f(v)$, the density of which is important in the thermodynamic limit, are located on the imaginary axis in the complex v-plane. So it is convenient to make the change of variable $v = is$. It is also useful to introduce another function $g(s) = f(is)/\cosh^{2n}(\frac{3s}{2})$. The differential equation for $g(s)$ is then

$$g'' + \left(\frac{9n(2n+1)}{2\cosh^2(\frac{3s}{2})} - 1\right) g = 0. \tag{16}$$

Let $g(s_0) = 0$. For large n we have in a small vicinity of s_0 an approximate equation $g'' + \omega_0^2 g = 0$. This equation describes a harmonic oscillator with frequency $\omega_0 = 3n/\cosh(\frac{3s_0}{2})$. The distance between nearest zeros is approximately $\Delta s = \frac{\pi}{\omega}$ and we obtain the following density function which describes the number of zeros per unit length:

$$\rho(s) = \frac{1}{\Delta s} = \frac{\omega}{\pi} = \frac{3n}{(\pi\cosh(\frac{3s}{2}))}.$$

We note that equation (16) has a history as rich as the BAE. Eckart [13] used the Schrodinger equation with bell-shaped potential $V(r) = -G/\cosh^2 r$ for phenomenological studies in atomic and molecular physics. Later it was used in chemistry, biophysics and astrophysics, to name just a few. For more recent references see, for example, [14].

After the completion of this letter, we were informed that in Baxter's review [8] he noticed the possibility of a simple eigenvalue of the transfer matrix for the XYZ model for the special value $\mu = \frac{\pi}{3}$ of the crossing parameter.

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